Non-integrability of homogeneous two-dimensional Hamiltonians with velocity-dependent potential

1988 J. Phys. A: Math. Gen. 21 3547

(http://iopscience.iop.org/0305-4470/21/18/011)

J. Phys. A: Math. Gen. 21 (1988) 3547-3557. Printed in the UK

Non-integrability of homogeneous two-dimensional Hamiltonians with velocity-dependent potential

D Roekaerts†‡ and H Yoshida§||

† Instituut voor Theoretische Fysica, Universiteit Leuven, B-3030 Leuven, Belgium
§ Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau, France

Received 2 November 1987, in final form 3 May 1988

Abstract. The non-integrability of classes of homogeneous two-dimensional Hamiltonian systems with a polynomial velocity-dependent potential is shown on the basis of Ziglin's theorem. An analytic expression for the trace of the relevant monodromy matrices is presented which fits the numerical data perfectly. An application is made to Fokker-Planck Hamiltonians with quadratic and with cubic drift terms.

1. Introduction

Given a class of two-dimensional Hamiltonian systems one is interested in identifying all completely integrable cases, i.e. all cases for which a second integral of motion exists [1]. A claim of completeness of the results can only be made if each case is either shown to be integrable or shown to be non-integrable. For the former, explicit construction of the second integral is sufficient, while for the latter a proof that some necessary condition following from the existence of a second integral is violated would be necessary.

The contribution to this ambitious programme that we present in this paper is a proof of non-integrability of large classes of Hamiltonian systems of the form

$$H(\boldsymbol{p},\boldsymbol{q})=\tfrac{1}{2}(p_1^2+p_2^2)+p_1A(q_1,q_2)+p_2B(q_1,q_2)+C(q_1,q_2). \qquad (1)$$

For general A, B and C, these systems describe particle motion in a transverse magnetic field—for $C=0$ they are associated with the weak noise limit of Fokker-Planck equations [2]. Their integrability has been studied in [2-5].

Sufficient conditions for non-integrability (i.e. non-existence of an additional analytic integral) are provided by Ziglin's theorem [6]. Originally it was applied to the motion of a rigid body around a fixed point [6]. Recently it has been applied to Hamiltonians of the form (1) with $A=B=0$ and $C(q_1,q_2)$: (i) a homogeneous potential [7, 8], (ii) some generalised Toda lattice [9], (iii) some perturbed Kepler potential [10], (iv) a non-homogeneous polynomial potential [11-15]. In physical applications, non-homogeneous systems are in general the most relevant. But in the method of [14] the results on homogeneous systems form an essential ingredient in the application to

‡ Present address: Koninklijke/Shell-Laboratorium Amsterdam, EP, Postbus 3003, NL-1003 AA Amsterdam, The Netherlands.
|| Present address: Department of Mathematics, Imperial College of Science and Technology, 180 Queen's Gate, London SW7 2BZ, UK.

non-homogeneous systems. Therefore, in the case that A and B in (1) are non-zero, it is natural to start with a study of homogeneous systems.

Let us assume that A, B and C in (1) are homogeneous polynomials of degree d, d and $2d$, respectively ($d \geq 2$). The application of Ziglin's theorem requires a knowledge of certain monodromy matrices associated with the normal variational equation (NVE) around straight line solutions. For the system under consideration here, in general no analytic formulae for the monodromy matrices are known. Therefore we have calculated them numerically. On the one hand, this leads to a purely numerical non-integrability proof on the basis of Ziglin's theorem, and on the other hand it turns out that the monodromy matrices are fitted perfectly by an analytic formula which is therefore conjectured to be rigorously exact.

The rest of this paper is organised as follows. In § 2, the NVE around straight line solutions is obtained. In § 3, Ziglin's theorem is recalled and its numerical implementation is explained. In § 4, our main results are presented. In § 5, two examples are given, and the appendix contains the FORTRAN program by which the numerical results of § 4 were obtained.

2. Variational equations around a straight line solution

By Hamilton's equations of motion

$$\mathrm{d}\boldsymbol{q}/\mathrm{d}t \equiv \partial \boldsymbol{H}/\partial \boldsymbol{p} \qquad \mathrm{d}\boldsymbol{p}/\mathrm{d}t = -\partial \boldsymbol{H}/\partial \boldsymbol{q} \tag{2}$$

with Hamiltonian (1) the equations of motion for $\boldsymbol{q}$ are written as

$$\ddot{q}_1 = \dot{q}_2 U(q_1, q_2) - W_{,1}(q_1, q_2) \tag{3}$$

$$\ddot{q}_2 = -\dot{q}_1 U(q_1, q_2) - W_{,2}(q_1, q_2) \tag{4}$$

where $W_{,1} := \partial W/\partial q_1$, etc, and $U(q_1, q_2)$ and $W(q_1, q_2)$ are gauge-invariant potentials:

$$U(q_1, q_2) = A_{,2}(q_1, q_2) - B_{,1}(q_1, q_2) \tag{5}$$

$$W(q_1, q_2) = C(q_1, q_2) - \tfrac{1}{2}\{A(q_1, q_2)\}^2 - \tfrac{1}{2}\{B(q_1, q_2)\}^2. \tag{6}$$

If $U = 0$, (1) can be transformed by a gauge transformation to a Hamiltonian with $A = B = 0$, the non-integrability of which was studied in [7, 8]. Given our assumptions on A, B and C, the general form of U and W is

$$U(q_1, q_2) = \sum_{k=0}^{d-1} u_k q_1^k q_2^{d-1-k} \tag{7}$$

$$W(q_1, q_2) = \sum_{k=0}^{2d} w_k q_1^k q_2^{2d-k} \tag{8}$$

with constants u_k, $k = 0, \ldots, d-1$ and w_k, $k = 0, \ldots, 2d$.

First we make a rotation of coordinates in the (q_1, q_2) plane so that $w_1 = 0$ in the new coordinate system, which is always possible. Then we *assume* also that $u_0 = 0$ in the same coordinates. This is a sufficient condition for (3) and (4) to have a straight line solution of the form

$$q_1 = 0 \tag{9}$$

with

$$\ddot{q}_2 = -w_{,2}(0, q_2) = -2dw_0 q_2^{2d-1}. \tag{10}$$

We now consider a particular solution of (10), namely

$$q_2(t) = c\phi(t) \tag{11}$$

with $\phi(t)$ the solution of

$$d^2\phi/dt^2 + \phi^{2d-1} = 0 \tag{12}$$

with initial conditions

$$\phi = 0 \qquad d\phi/dt = -1/\sqrt{d} \qquad \text{at } t = 0 \tag{13}$$

and ($w_0 \neq 0$ is assumed)

$$c = \left(\frac{1}{2dw_0}\right)^{1/(2d-2)}. \tag{14}$$

Because of the scale invariance of the system (2)-(8) the non-integrability at the particular non-zero energy associated with (11) implies the non-integrability at any non-zero energy. Therefore it is sufficient to consider the particular solution (11).

Since $\phi(t)$ is the inverse function of

$$t = \sqrt{d} \int dz/(1 - z^{2d})^{1/2} \tag{15}$$

$\phi(t)$ has, in the complex t plane, the following independent periods:

$$\begin{aligned} T_n &= 4\exp[i\pi(n-1)/d]\sqrt{d} \int_0^1 dz/(1 - z^{2d})^{1/2} \\ &= 2(\pi/d)^{1/2} \exp[i\pi(n-1)/d]\Gamma(1/2d)/\Gamma(1/2d + \tfrac{1}{2}) \end{aligned} \tag{16}$$

($n = 1, 2, \ldots, d$).

Next, linearising (3) and (4) around the straight line solution $q_1 = 0$, one obtains the variational equation ($\xi_1 = \delta q_1$, $\xi_2 = \delta q_2$)

$$\ddot{\xi}_1 + (\varepsilon\phi^{2d-2} - i\gamma\dot{\phi}\phi^{d-2})\xi_1 = 0 \tag{17}$$

and

$$\ddot{\xi}_2 + (2d-1)\phi^{2d-2}\xi_2 = 0 \tag{18}$$

with

$$\gamma = u_1/(-2dw_0)^{1/2} \qquad \varepsilon = w_2/dw_0. \tag{19}$$

The equation for ξ_1 (17) is called the normal variational equation (NVE), since it describes the variation normal to the given straight line solution $q_1 = 0$. The parameters entering the NVE are d, characterising the degree of the potential, and ε and γ defined by (19). It should be remarked that γ and ε are independent of u_k, $k > 1$, and w_k, $k > 2$.

In the next section we show how, using Ziglin's theorem for every d, values of γ and ε can be found for which the Hamiltonian system is non-integrable.

3. Ziglin's theorem

The NVE (17) is a Hill equation with multiple periods $T_1, T_2, \ldots, T_d$. To each period T_n, $1 \leq n \leq d$, is associated a monodromy matrix $M(T_n)$. The set of all monodromy matrices forms a group, called the monodromy group. A monodromy matrix M is defined to be non-resonant when the eigenvalues $(\rho, 1/\rho)$ are not roots of unity. One has that $|\mathrm{Tr}\, M| > 2$ implies that M is non-resonant. The form of Ziglin's theorem that we use in this paper is as follows.

Theorem [6]. If the monodromy group associated with the NVE belonging to the straight line solution $q_1 = 0$ contains two non-resonant monodromy matrices which do not commute, then the Hamiltonian system cannot possess an additional integral $\Phi(q, p) =$ constant which is analytic (holomorphic), at least in the neighbourhood of the given straight line.

When $\gamma = 0$, the monodromy matrices of the NVE (17) have been obtained explicitly via a transformation of the NVE into the Gauss hypergeometric equation. The result of [7] is summarised as follows: Tr $M(T_n)$ is independent of the period T_n and the explicit expression is Tr $M(T_n) = f_{2d}(\varepsilon)$, where

$$f_{2d}(\varepsilon) = 4\cos^2\{(\pi/2d)[(d-1)^2 + 4d\varepsilon]^{1/2}\}/\sin^2(\pi/2d) - 2. \tag{20}$$

Furthermore, $M(T_1)$ and $M(T_2)$ only commute when $f_{2d}(\varepsilon) = \pm 2$. Thus, if ε is in the region

$$S_{2d} = \{\varepsilon < 0, 1 < \varepsilon < 2d - 1, 2d + 2 < \varepsilon < 6d - 2, \ldots\} \tag{21}$$

such that $f_{2d}(\varepsilon) > 2$, then two monodromy matrices $M(T_1)$ and $M(T_2)$ are both non-resonant and non-commuting. By Ziglin's theorem, this implies the non-integrability of the system and region (21) is called the non-integrability region.

When $\gamma \neq 0$ we have no explicit expression of the monodromy matrices. Nevertheless in this situation Ziglin's theorem can be applied on the basis of numerical data. The monodromy matrices $M(T_n)$, $1 \leq n \leq d$ can be obtained numerically as follows. Consider two independent solutions of the NVE (17), $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ with initial conditions

$$\xi^{(1)}(0) = 1 \qquad \dot{\xi}^{(1)}(0) = 0 \qquad \xi^{(2)}(0) = 0 \qquad \dot{\xi}^{(2)}(0) = 1 \tag{22}$$

and for every period T_n, $1 \leq n \leq d$, integrate the NVE (17) numerically along the straight path from 0 to T_n in the complex t plane (using, e.g., the fourth-order complex Runge–Kutta method). At the end of each integration path one gets the monodromy matrix

$$M_n \equiv M(T_n) = \begin{pmatrix} \xi^{(1)}(T_n) & \xi^{(2)}(T_n) \\ \dot{\xi}^{(1)}(T_n) & \dot{\xi}^{(2)}(T_n) \end{pmatrix}. \tag{22}$$

When it is found that $|\mathrm{Tr}\, M_n| > 2$ it can be concluded that M_n is non-resonant. For $|\mathrm{Tr}\, M_n| \leq 2$ non-resonant monodromy matrices cannot be identified numerically. Whether or not two monodromy matrices M_k and M_l commute can be decided on the basis of the numerical value of the following norm of the commutator:

$$\|[M_k, M_l]\| = \sum_{i,j=1}^{2} |(M_k M_l - M_l M_k)_{ij}|. \tag{24}$$

When it is significantly non-zero, one can say that M_k and M_l do not commute.

Finally, if one finds two non-resonant monodromy matrices that do not commute one can conclude that the system is non-integrable. In order to find for any fixed d the non-integrability region in the (ε, γ) plane, generalising the non-integrability region S_{2d} on the line $\gamma = 0$, this procedure is to be followed for all (ε, γ) on a sufficiently dense grid in this plane.

In the next section the results we obtained using this method are given.

4. Main results—numerical facts

By using the FORTRAN program given in the appendix and straightforward extensions of it, we computed monodromy matrices M_n, $n=1,2,\ldots,d$, for $d=2,3,4,5,6$ at square grid points on the (ε, γ) plane and, by the procedure explained in the previous section, obtained the non-integrability region. We also found analytical formulae which perfectly fit the numerical data for the traces of the monodromy matrices and for the curves where two monodromy matrices commute. We checked the formulae up to $d=10$. Until now we have not found a rigorous proof of these formulae, so we present them as numerical facts.

A systematic difference occurs between the cases with d even and with d odd. A special role is played by certain straight lines and parabolas in the (ε, γ) plane which we introduce first.

Definition.

(i) for d even

$$P^d=\{(\varepsilon,\gamma)|\varepsilon-\gamma^2/(kd+1)^2-(kd+1)^2/(4d)+(d-1)^2/(4d)=0,\ k \text{ odd integer}\} \quad (25)$$

$$K_+^d=\{(\varepsilon,\gamma)|\varepsilon+\gamma/\sqrt{d}-dk^2/4+(d-1)^2/(4d)=0,\ k \text{ odd integer}\} \quad (26)$$

$$K_-^d=\{(\varepsilon,\gamma)|\varepsilon-\gamma/\sqrt{d}-dk^2/4+(d-1)^2/(4d)=0,\ k \text{ odd integer}\}. \quad (27)$$

(ii) For d odd

$$Q^d=\{(\varepsilon,\gamma)|\varepsilon+\gamma/k^2d^2-k^2d/4+(d-1)^2/(4d)=0,\ k \text{ odd integer}\} \quad (28)$$

$$L_+^d=\{(\varepsilon,\gamma)|\varepsilon+\gamma/\sqrt{d}-(kd+1)^2/(4d)+(d-1)^2/(4d)=0,\ k \text{ odd integer}\} \quad (29)$$

$$L_-^d=\{(\varepsilon,\gamma)|\varepsilon-\gamma/\sqrt{d}-(kd+1)^2/(4d)+(d-1)^2/(4d)=0,\ k \text{ odd integer}\}. \quad (30)$$

For the trace of the monodromy matrices M_n we obtain the following result.

Fact 1. Let

$$s_+=[(d-1)^2+4d(\varepsilon+\gamma/\sqrt{d})]^{1/2} \quad (31)$$

$$s_-=[(d-1)^2+4d(\varepsilon-\gamma/\sqrt{d})]^{1/2} \quad (32)$$

then $\operatorname{Tr} M_n$, $n=1,2,\ldots,d$ is real and independent of n and given by

$$\operatorname{Tr} M_n = F_{2d}(\varepsilon,\gamma) \quad (33)$$

where (i) when d is even

$$F_{2d}(\varepsilon,\gamma)=2+4\{\cos(\pi/d)+\cos[\pi(s_++s_-)/2d]\} \times\{\cos(\pi/d)+\cos[\pi(s_+-s_-)/2d]\}/\sin^2(\pi/d) \quad (34)$$

or (ii) when d is odd

$$F_{2d}(\varepsilon,\gamma)=-2+4\cos^2[\pi(s_++s_-)/4d]\cos^2[\pi(s_+-s_-)/4d]/\sin^2(\pi/2d). \quad (35)$$

Properties which (34) and (35) have in common are

$$F_{2d}(\varepsilon,0)=f_{2d}(\varepsilon) \quad (36)$$

(i.e. the necessary compatibility relation with the known result for $\gamma=0$ is satisfied)

$$F_{2d}(\varepsilon,\gamma)=F_{2d}(\varepsilon,-\gamma) \quad (37)$$

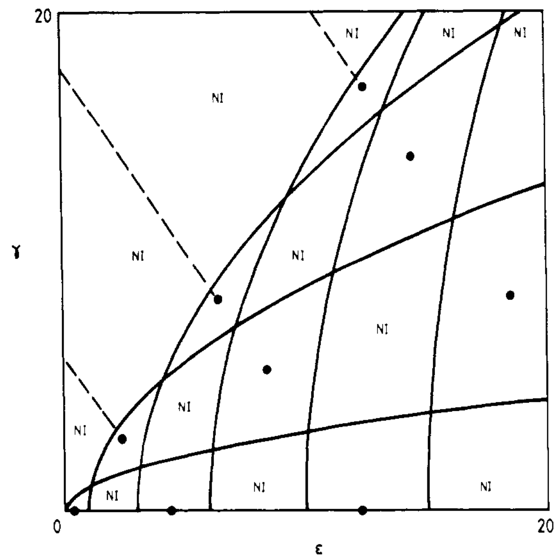

Figure 1. Non-integrability diagram belonging to the NVE (17) for $d=2$. On the parabolas $F_4(\varepsilon,\gamma)=2$ and on the isolated points $F_4(\varepsilon,\gamma)=-2$. The non-integrability region, indicated by NI, is the region where $F_4(\varepsilon,\gamma)>2$ minus the broken lines (see the text).

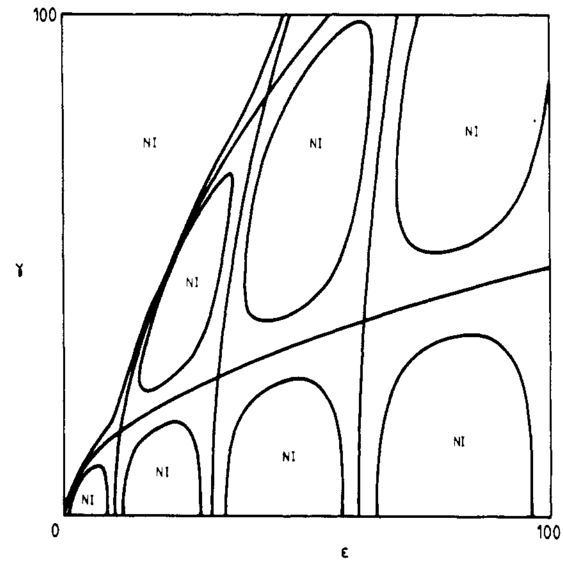

Figure 2. Non-integrability diagram belonging to the NVE (17) for $d=3$. On the parabolas $F_6(\varepsilon,\gamma)=-2$ and on the other curves $F_6(\varepsilon,\gamma)=+2$. The non-integrability region, indicated by NI, is the region where $F_6(\varepsilon,\gamma)>2$.

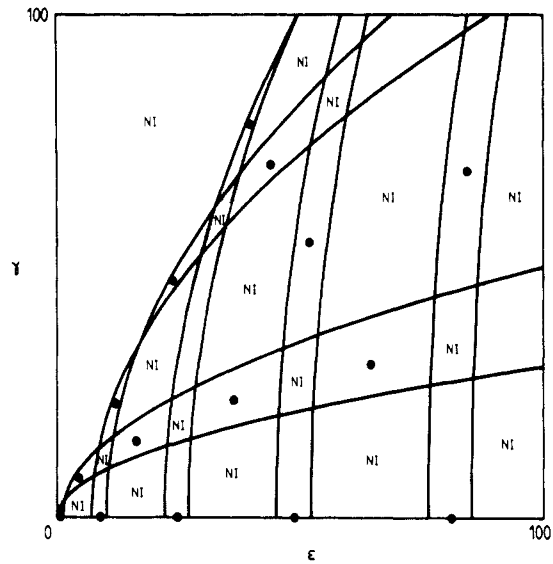

Figure 3. The non-integrability diagram belonging to the NVE (17) for $d = 4$. On the parabolas $F_8(\varepsilon, \gamma) = +2$ and on the isolated points $F_8(\varepsilon, \gamma) = -2$. The non-integrability region is the region where $F_8(\varepsilon, \gamma) > 2$.

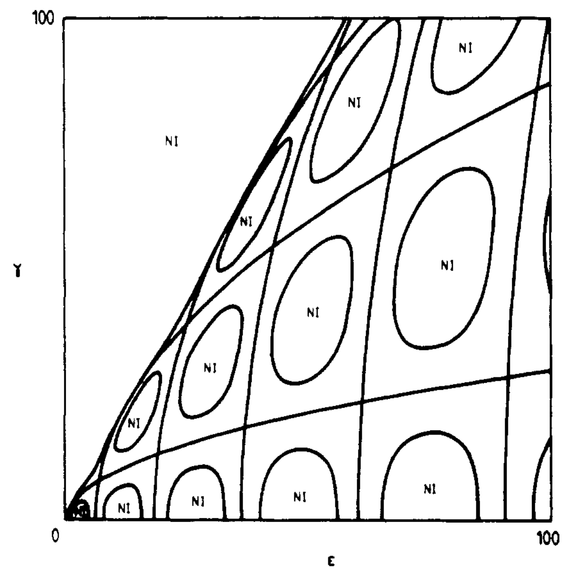

Figure 4. Non-integrability diagram belonging to the NVE (17) for $d = 5$. On the parabolas $F_{10}(\varepsilon, \gamma) = -2$ and on the other curves $F_{10}(\varepsilon, \gamma) = +2$. The non-integrability region is the region where $F_{10}(\varepsilon, \gamma) > 2$.

and

$$\min_{(\varepsilon,\gamma)} F_{2d}(\varepsilon, \gamma) = -2. \tag{38}$$

Further important properties of $F_{2d}(\varepsilon, \gamma)$ can be seen in figures 1-4. There the sets where $F_{2d}(\varepsilon, \gamma) = \pm 2$ are drawn. In figure 1, $d = 2$ and ε and γ range from 0 to 20. In figures 2, 3 and 4, $d = 3, 4$ and 5, respectively, and ε and γ range from 0 to 100. One has that:

(i) when d is even, $F_{2d}(\varepsilon, \gamma) = 2$ on the parabolas P^d given by (25) and $F_{2d}(\varepsilon, \gamma) = -2$ in the points where the two families of straight lines K_+^d and K_-^d given by (26) and (27) intersect;

(ii) when d is odd the curves on which $F_{2d}(\varepsilon, \gamma) = 2$ are not represented by a simple function (see figures 2 and 4) and $F_{2d}(\varepsilon, \gamma) = -2$ on the parabolas Q^d given by (28).

In the region such that $F_{2d}(\varepsilon, \gamma) > 2$ we have non-resonant monodromy matrices $M_1, M_2, \ldots, M_d$. In particular the half-plane $\varepsilon < 0$ belongs to this region.

It should be remarked that it was also found numerically that elements of the monodromy group other then $M_1, \ldots, M_d$ never have absolute values of trace greater then two when $\mathrm{Tr}\, M_n < 2$ and therefore cannot be used to provide a non-integrability proof in the region where $F_{2d}(\varepsilon, \gamma) \leq 2$.

Next we proceed to the commutation properties of the monodromy matrices $M_1, \ldots, M_d$.

In general, the set of points where M_k and M_l $(1 \leq k \leq d)$ commute depends on k and l. With Ziglin's theorem in mind an optimal result would be that M_k and M_l never commute when they are non-resonant. For d odd this optimal result is already obtained choosing $k = 1, l = 2$ (see below), whereas for d even this choice is not optimal because the set of points in the (ε, γ) plane where M_1 and M_2 commute partially overlaps with the region where M_1 and M_2 are non-resonant. Therefore the commutator $[M_1, M_3]$ was considered for d even and greater than two, and found to be optimal. More precisely the following results were obtained.

Fact 2.

(i) When $d = 2$, M_1 and M_2 only commute on the sets P^2, K_+^2 and K_-^2 given by (25), (26) and (27), respectively.

(ii) When d is even and greater than two, M_1 and M_3 only commute on the set P^d, given by (25), and in the points where the sets K_+^d and K_-^d given by (26) and (27) intersect.

(iii) When d is odd, M_1 and M_2 only commute on the set Q^d given by (28) and in the points where the sets L_+^d and L_-^d given by (29) and (30) intersect. In fact, these intersection points fall into two classes as follows: let $k_+ = 2p_+ + 1$ and $k_- = 2p_- + 1$ be the two odd integers characterising a line belonging to L_+^d and L_-^d, respectively. When p_+ and p_- are not both even or not both odd the intersection point belongs to Q^d and when p_+ and p_- are both even or both odd the intersection point lies on the curve where $F_{2d}(\varepsilon, \gamma) = 2$.

Combining fact 1 and fact 2 with Ziglin's theorem we can conclude the following statements.

(i) When $d = 2$ the non-integrability region in the (ε, γ) plane is the region where $F_4(\varepsilon, \gamma) > 2$ minus its intersection with the families of straight lines K_+^2 and K_-^2 given by (25) and (26), respectively.

(ii) When $d > 2$ the non-integrability region in the (ε, γ) plane is the region where $F_{2d}(\varepsilon, \gamma) > 2$.

In figures 1-4 the non-integrability region is indicated by NI.

Because of the high numerical accuracy and the consistency of all results, the authors strongly expect that facts 1 and 2 will be rigorously proved in the near future.

5. Examples

Any subclass of the models defined by (7) and (8) with, eventually in a rotated frame, $u_0 = 0$, $w_0 \neq 0$, and $w_1 = 0$ could be taken as an example. Interesting subclasses are those for which integrability has been studied before or that appear in a physical context.

As a first example we consider a model with $d = 2$, defined by

$$H = \tfrac{1}{2}(p_1^1 + p_2^2) + p_1 q_1 q_2 + p_2(aq_1^2 + bq_2^2) \tag{39}$$

with constants a and b. When $a = \frac{1}{2}$, the function U in (5) and (6) vanishes and the system is related by a gauge transformation to a one-parameter class of quartic scalar potentials and is known to be integrable for

$$b = -1, -\tfrac{1}{2}, 1 \text{ and } 2. \tag{40}$$

When $a \neq \frac{1}{2}$, (39) was shown to be integrable in [5] for

$$(a, b) = (\tfrac{1}{8}, 2), (-\tfrac{1}{4}, \tfrac{1}{2}), (\tfrac{1}{32}, 2), (0, 1) \text{ and } (0, -1). \tag{41}$$

Applying the results of the previous section, non-integrability is shown for all values of a and b $(b \neq 0)$ such that $(\varepsilon(a, b), \gamma(a, b))$ with

$$\varepsilon = (1 + 2ab)/2b^2 \qquad \gamma = (1 - 2a)/\sqrt{2b} \tag{42}$$

does not belong to (26) or (27) with $d = 2$, and $F_4(\varepsilon, \delta) > 2$ in (34). In particular, non-integrability is shown for

$$(a, b) = (\tfrac{3}{8}, -2) \text{ and } ((18 + m)(18 - m)/(16m^2), 4) \tag{43}$$

with $m = 1$ or $m > 18$, which are values for which the model passes the second step of Painlevé analysis and has rational Kowalevski exponents (see equation (32) in [5]).

As a second example we consider the Fokker-Planck Hamiltonians with cubic drift terms [2], $(d = 3)$, defined by (1) with $C = 0$ and

$$A = a_1 q_1^3 + a_2 q_1^2 q_2 + a_3 q_1 q_2^2 + a_4 q_2^3 \tag{44}$$

$$B = b_1 q_1^3 + b_2 q_1^2 q_2 + b_3 q_1 q_1^2 + b_4 q_2^3 \tag{45}$$

where a_i, b_i are constants. By (5) and (6) the potentials U and W are computed. The existence of a straight line solution $q_1 = p_1 = 0$ implies $u_0 = 3a_4 - b_3 = 0$ and $w_1 = -a_3 a_4 - b_3 b_4 = 0$. If also $w_0 = -\frac{1}{2}(a_4^2 + b_4^2) \neq 0$ the results of the previous section can be applied with

$$\gamma = 2(a_3 - b_2)/[3(a_4^2 + 2b_4^2)]^{1/2} \tag{46}$$

$$\varepsilon = (a_3^2 + b_3^2 + 2a_2 a_4 + 2b_2 b_4)/[3(a_4^2 + b_4^2)]. \tag{47}$$

By the results of § 4, the existence of a second integral for a model in this class is excluded in the range of parameters where $F_6(\varepsilon, \gamma) > 2$. Finally we remark that integrable cases are given in the appendix of [2]. Taking into account all straight line solutions it can be shown that, for the model studied in that appendix, the results are complete, namely, the model is non-integrable except for the cases which were shown to be integrable.

Acknowledgments

One of us (DR) acknowledges support by the National Fund for Scientific Research (Belgium) as senior research assistant and by the Alexander von Humboldt Foundation for a stay at the Institute of Physics of the University of Essen where part of this work was done.

Appendix

```
C***********************************************************************
C CALCULATION OF TRACE OF MONODROMY MATRICES                           *
C PARAMETERS TO BE CHOSEN ARE DEGREE D AND EPSILON AND GAMMA           *
C***********************************************************************
      PROGRAM MAIN
      IMPLICIT REAL*8(A-B,D-H,O-Z),COMPLEX*16(C)
      COMMON EPS,GAM,PI,IDEG,NSTEP
      NSTEP=50
      GOTO 2
 1    WRITE(6,*)'YOUR CHOICE ? 0=STOP, 1=NEXT'
      READ(5,*)IFLAG
      IF(IFLAG)999,999,2
 2    WRITE(6,*)'DEGREE D ? (INTEGER),EPSILON AND GAMMA ? (REAL)'
      READ(5,*)IDEG,EPS,GAM
      WRITE(6,*)'TRACE OF MONODROMY MATRICES MI, I=1,...,D'
      PI=4.D0*DATAN(1.D0)
      DX=DCOS(PI/DFLOAT(IDEG))
      DY=DSIN(PI/DFLOAT(IDEG))
      COMEGA=CMPLX(DX,DY)
      DO 4 KK=1,IDEG
      CTINIT=(1.D0,0.0D0)
      CTFINA=CTINIT+(4.0D0,0.0D0)*COMEGA**(KK-1)
      CALL TRACE(CTINIT,CTFINA,CTRACE)
   4  WRITE(6,*)CTRACE
      GO TO 1
 999  STOP
      END

C*************************************************************
C SUBROUTINE TRACE CALCULATES TRACE OF MONODROMY MATRIX (CTRACE)*
C FOR STRAIGHT LINE IN COMPLEX TIME PLANE FROM CTINIT TO CTFINA *
C*************************************************************
      SUBROUTINE TRACE(CTINIT,CTFINA,CTRACE)
      IMPLICIT REAL*8(A-B,D-H,O-Z),COMPLEX*16(C)
      DIMENSION CX(6)
      COMMON EPS,GAM,PI,IDEG,NSTEP
      EXTERNAL CEQ1
      DEG=DFLOAT(IDEG)
      FACT=DSQRT(PI/DEG)/2.D0
      UNIT=FACT*DGAMMA(0.5D0/DEG)/DGAMMA(0.5D0/DEG+.5D0)
      CT=CTINIT
      CX(1)=0.D0
      CX(2)=-1.D0/SQRT(DEG)
      CX(3)=1.D0
      CX(4)=0.D0
      CX(5)=0.D0
      CX(6)=1.D0
      CDELT=(CTINIT-CTFINA)/NSTEP*UNIT
      DO 20 J=1,NSTEP
      CALL CRUNGE(6,CEQ1,CT,CX,CDELT)
 20   CONTINUE
      CTRACE=CX(3)+CX(6)
      RETURN
      END
```

```
C***********************************************************
C COMPLEX FUNCTION CEQ1 CONTAINS THE EQUATIONS THAT HAVE TO *
C BE INTEGRATED                                             *
C***********************************************************

      COMPLEX FUNCTION CEQ1*16(I,CX)
      IMPLICIT REAL*8(A-B,D-H,O-Z),COMPLEX*16(C)
      DIMENSION CX(6)
      COMMON EPS,GAM,PI,IDEG,NSTEP
      CGAM=(0.D0,1.D0)*GAM
      GO TO (10,20,30,40,50,60),I
 10   CEQ1=CX(2)
      RETURN
 20   CEQ1=     -CX(1)**(2*IDEG-1)
      RETURN
 30   CEQ1=   CX(4)
      RETURN
 40   IF(IDEG.GT.2)GOTO 42
 41   CEQ1=( EPS*CX(1)**2+CGAM*CX(2))*CX(3)
      RETURN
 42   CEQ1=(-EPS*CX(1)**(2*IDEG-2)+CGAM*CX(1)**(IDEG-2)*CX(2))*CX(3)
      RETURN
 50   CEQ1=CX(6)
      RETURN
 60   IF(IDEG.GT.2)GOTO 62
 61   CEQ1=(-EPS*CX(1)**2+CGAM*CX(2))*CX(5)
      RETURN
 62   CEQ1=(-EPS*CX(1)**(2*IDEG-2)+CGAM*CX(1)**(IDEG-2)*CX(2))*CX(5)
      RETURN
      END

C*********************************************************************
C SUBROUTINE CRUNGE IS A COMPLEX FOURTH ORDER RUNGE-KUTTA SCHEME      *
C FOR INTEGRATION OF N EQUATIONS FROM T TO T+DT                       *
C*********************************************************************

      SUBROUTINE CRUNGE(N,F,T,X,DT)
      IMPLICIT COMPLEX*16(A-H,O-Z)
      DIMENSION X(6),X1(6),X2(6),X3(6),D(4,6)
      DO 10 I=1,N
      D(1,I)=F(I,X)*DT
 10   X1(I)=X(I)+D(1,I)/2.0D0
      DO 20 I=1,N
      D(2,I)=F(I,X1)*DT
 20   X2(I)=X(I)+D(2,I)/2.0D0
      DO 30 I=1,N
      D(3,I)=F(I,X2)*DT
 30   X3(I)=X(I)+D(3,I)
      T=T+DT
      DO 40 I=1,N
      D(4,I)=F(I,X3)*DT
 40   X(I)=X(I)+(D(1,I)+2.D0*D(2,I)+2.D0*D(3,I)+D(4,I))/6.D0
      RETURN
      END
```

References

[1] Hietarinta J 1987 *Phys. Rep.* **147** 87

[2] Graham R, Roekaerts D and Tél T 1985 *Phys. Rev.* A **31** 3364

[3] Dorizzi B, Grammaticos B, Ramani A and Winternitz P 1985 *J. Math. Phys.* **26** 3070

[4] Hietarinta J 1985 *J. Math. Phys.* **26** 1970

[5] Roekaerts D and Schwarz F 1987 *J. Phys. A: Math. Gen.* **20** L127

[6] Ziglin S L 1983 *Funct. Anal. Appl.* **16** 181, **17** 6

[7] Yoshida H 1986 *Physica* **21D** 163

[8] Yoshida H 1988 *Physica* **29D** 128

[9] Yoshida H, Ramani A, Grammaticos B and Hietarinta J 1987 *Physica* **144A** 310

[10] Yoshida H 1987 *Phys. Lett.* **120A** 388

[11] Ito H 1985 *Kodai Math. J.* **8** 120

[12] Ito H 1987 *J. Appl. Math. Phys.* **38** 459

[13] Rod D L 1986 *Preprint. On a theorem of Ziglin in Hamiltonian dynamics*

[14] Yoshida H 1988 *Commun. Math. Phys.* **116** 529

[15] Yoshida H, Ramani A and Grammaticos B 1988 *Physica* **30D** 151